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Scattering with a periodically kicked interaction and cyclic states

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Abstract. A quantum-mechanical particle kicked by a rank 1 perturbation is a solvable model of a scattering system with a time periodic Hamiltonian. We study its spectral properties and compute explicitly the scattering quantities. Below a critical value, there are certain ranges of periods for which the system has stable states under the time evolution (cyclic states). These cyclic states lose their stability as the period increases and may be transformed into resonances. This is an example of the general phenomenon of stabilization of quantum states under a high-frequency perturbation.

1. Introduction

Scattering of a quantum-mechanical particle by a short-range time periodic potential plays an important role in a variety of problems ranging from tunnelling through a modulated barrier to atoms in laser fields. Although the general formalism is well developed [1, 2], it is notoriously difficult to perform explicit calculations of scattering probabilities. In this paper, we present a solvable model, the periodically kicked potential, that may serve as an illustrative paradigm of several typical phenomena.

We consider a system described by a free Hamiltonian H_0 kicked by a rank 1 interaction with period T. The total Hamiltonian reads formally

$$H(t) = H_0 + \lambda f(t)|\varphi\rangle\langle\varphi| \tag{1}$$

with

$$f(t) = T \sum_{n=-\infty}^{\infty} \delta(t - nT). \tag{2}$$

The free Hamiltonian $H_0 = \int E \, \mathrm{d}P(E)$ is a self-adjoint operator on a Hilbert space $\mathcal{H}$ with simple absolutely continuous spectral measure $\mathrm{d}P(E)$ supported in the interval $[0, E_0]$. For convenience, we also sometimes use the formal Dirac notation $\mathrm{d}P(E) = |E\rangle\langle E| \, \mathrm{d}E$ with $\langle E|E'\rangle = \delta(E - E')$, E, E' in $[0, E_0]$, and the convention that $|E\rangle\langle E|$ vanishes outside of the spectrum of H_0. The normalized vector $|\varphi\rangle$ in $\mathcal{H}$ is represented by the function $\varphi(E)$ in the spectral representation of H_0, and $\lambda|\varphi\rangle\langle\varphi|$ is the corresponding rank 1 perturbation with coupling constant λ. Throughout the paper we assume that $\varphi(E)$ is absolutely continuous and does not vanish on the spectrum of H_0, except possibly at its extremities $E = 0$ and $E = E_0$. By convention, $\varphi(E)$ is equal to zero outside of $[0, E_0]$. The kick function $f(t)$ is normalized to have its time average over one period equal to 1, $\frac{1}{T}\int_0^T f(t) \, \mathrm{d}t = 1$.

The evolution operator $U(t)$ associated with the formal Hamiltonian (1) acts as follows. For $0 < t < T$, the system evolves freely†

$$U(t) = U_0(t) = \mathrm{e}^{-\mathrm{i}H_0 t} \qquad 0 < t < T. \tag{3}$$

Just after the first kick, that occurs at time T, one has

$$U(T_+) = \lim_{\tau \to 0} U(T+\tau) = \mathrm{e}^{-\mathrm{i}H_0 T}\mathrm{e}^{-\mathrm{i}\lambda T|\varphi\rangle\langle\varphi|} \tag{4}$$

and the evolution continues perodically in this way. We denote simply the free evolution over one period by $U_0 = \mathrm{e}^{-\mathrm{i}H_0 T}$ and the corresponding total evolution (the Floquet operator) by

$$U = U(T_+) = U_0(I + \mu|\varphi\rangle\langle\varphi|) \tag{5}$$

where

$$\mu = \mathrm{e}^{-\mathrm{i}\lambda T} - 1. \tag{6}$$

The simple form (5) results from the fact that $|\varphi\rangle\langle\varphi|$ is a projector.

The spectral properties of U have been studied by Combescure [3] when H_0 has a pure point spectrum, and there is a large literature on kicked Hamiltonians with discrete spectrum (kicked rotator, kicked oscillator) corresponding to confining potentials (see for instance [4]). However, to our knowledge, less attention has been paid to the case when H_0 has a continuous spectrum allowing for scattering processes.

The spectral theorem for unitary operators yields

$$U = \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}\theta}\,\mathrm{d}F(\theta) \tag{7}$$

$$U_0 = \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}\theta}\,\mathrm{d}F_0(\theta) \tag{8}$$

where $\mathrm{d}F(\theta)$ and $\mathrm{d}F_0(\theta)$ are spectral measures on the unit circle‡. The spectral measures of U_0 and H_0 are related by

$$\mathrm{d}F_0(\theta) = \sum_n \left|\frac{\theta+2\pi n}{T}\right\rangle\left\langle\frac{\theta+2\pi n}{T}\right| \frac{\mathrm{d}\theta}{T} \qquad 0 \leqslant \theta < 2\pi. \tag{9}$$

This results from the identity

$$\begin{aligned} U_0 &= \int_0^{E_0} \mathrm{e}^{-\mathrm{i}ET}|E\rangle\langle E|\,\mathrm{d}E = \int_0^{\theta_0} \mathrm{e}^{-\mathrm{i}\theta}\left|\frac{\theta}{T}\right\rangle\left\langle\frac{\theta}{T}\right|\frac{\mathrm{d}\theta}{T} \\ &= \sum_n \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}\theta}\left|\frac{\theta+2\pi n}{T}\right\rangle\left\langle\frac{\theta+2\pi n}{T}\right|\frac{\mathrm{d}\theta}{T} \qquad \theta_0 = E_0 T. \end{aligned} \tag{10}$$

It is useful to introduce also the frequency $\omega = \frac{2\pi}{T}$ and the quasi-energy ϵ by setting for any E

$$E = n\omega + \epsilon \tag{11}$$

† We set the Planck constant equal to 1.
‡ The sign of the phase factors in (7) and (8) is opposite to that appearing in the usual formulation of the spectral theorem for unitary operators. This is to match the standard definition of the quantum mechanical evolution (3). Here we call spectrum of U (U_0) the support of the spectral measure $\mathrm{d}F(\theta)$ ($\mathrm{d}F_0(\theta)$) with the present sign convention.

where n is the integer part of $\frac{E}{\omega}$, and $0 \leqslant \epsilon < \omega$; in particular $E_0 = n_0\omega + \epsilon_0$. In terms of the quasi-energy variable $\epsilon = \frac{\theta}{T}$, $0 \leqslant \theta < 2\pi$, one has

$$\mathrm{d}F_0(\epsilon) = \sum_n |\epsilon + n\omega\rangle\langle\epsilon + n\omega|\,\mathrm{d}\epsilon. \tag{12}$$

The period $T_0 = \frac{2\pi}{E_0}$ will be called the critical period, corresponding to the critical frequency $\omega_0 = E_0$. For $T < T_0$, one has $\epsilon_0 = E_0,\ n_0 = 0$, and only the term $n = 0$ occurs in (9); hence U_0 has absolutely continuous spectrum on the sector $[0, \theta_0 = E_0 T]$ of the unit circle, $\theta_0 < 2\pi$. If $T > T_0$, there is no gap: one sees in (10) that the spectrum of H_0 wraps the unit circle n_0 times. Thus U_0 has absolutely continuous spectrum on the whole circle with multiplicity $n_0 + 1$ if $0 \leqslant \theta \leqslant \epsilon_0 T$, and n_0 if $\epsilon_0 T < \theta < 2\pi$.

The main phenomenon that will be described in the remainder of the paper is as follows. Since the Floquet operator U differs from U_0 by a finite rank perturbation (see (5)), general stability theorems [5] ensure that it will also have an absolutely continuous spectrum in $[0, \theta_0]$; however, for $T < T_0$, U may acquire an additional discrete point $\mathrm{e}^{-\mathrm{i}\alpha_c}$ with eigenphase α_c in the sector $(\theta_0, 2\pi)$. The corresponding eigenvector ψ_c, also called cyclic state, is stable in the sense that it is left invariant by the evolution over one period

$$U\psi_c = \mathrm{e}^{-\mathrm{i}\alpha_c}\psi_c \qquad \alpha_c = \epsilon_c T \qquad \theta_0 < \alpha_c < 2\pi. \tag{13}$$

As T increases, α_c moves on the circle and for a certain period $T_c < T_0$ it may reach the threshold of the continuous spectrum either at $\theta = 0$ or at $\theta = \theta_0$. For $T > T_c$, the cyclic state loses its stability, and if the coupling function φ has suitable properties, it may be transformed into a long-living resonance seen in the scattering amplitude. No cyclic state can survive if the period becomes larger than T_0 because then the spectrum of U will be absolutely continuous on the whole circle. The period T_0 is said to be critical since in this model, no state can remain stable under the time evolution beyond the closure of the gap at T_0.

This situation is an explicit example of stabilization under high-frequency motion (see for instance [6] in a classical context and [7, 8] in atomic physics) which can be understood here as follows. In the high-frequency limit, it is known that a system submitted to a time periodic potential is governed by an effective static Hamiltonian $\bar{H}$ obtained by averaging the interaction over one period ([9] and references there). In our case we have

$$\bar{H} = H_0 + \lambda|\varphi\rangle\langle\varphi| \tag{14}$$

as follows also from (4) and the Trotter product formula: the evolution

$$U^N = \left(\exp\left(-\mathrm{i}\frac{H_0}{N}\right)\exp\left(-\mathrm{i}\frac{\lambda|\varphi\rangle\langle\varphi|}{N}\right)\right)^N$$

after N periods of duration $T = \frac{1}{N}$ tends to $\exp(-\mathrm{i}\bar{H})$ as $N \to \infty$. It turns out that when $T > 0$ is sufficiently small, the cyclic state ψ_c for the time-dependent evolution (1) with quasi-energy ϵ_c is close to an eigenvector of $\bar{H}$ and ϵ_c reduces to the corresponding eigenvalue of $\bar{H}$ as $T \to 0$. So in the range of periods $0 < T < T_0$, the time periodic evolution inherits the properties of that generated by a static Hamiltonian: an eigenvector of $\bar{H}$ is turned into a cyclic state, but this stability does not extend for $T > T_0$.

We construct the cyclic state in section 2. In section 3 we give general expressions for the wave operators in terms of the resolvent of the Floquet operator (the equivalent of the usual Lippman–Schwinger equations) and apply them to our model in section 4. We find simple formulae for the scattering matrix: we verify there the validity of Levinson's theorem and discuss the problem of resonances. Concluding remarks on possible generalizations are presented in section 5.

2. Existence of cyclic states

Here we follow closely the arguments of section 2 of [3] where conditions for the existence of point spectrum of U are given.

For any complex number z, $|z| \neq 1$, we introduce the resolvent $R(z) = (U - z)^{-1}$ of the Floquet operator. The solvability of the model relies on the fact that one can express $R(z)$ explicitly in terms of the free resolvent $R_0(z) = (U_0 - z)^{-1}$. The first resolvent identity yields

$$R(z) = R_0(z) - \mu|\varphi\rangle\langle\varphi|R(z) - z\mu R_0(z)|\varphi\rangle\langle\varphi|R(z) \tag{15}$$

which implies for any ψ

$$\langle\varphi|R(z)|\psi\rangle = \frac{\langle\varphi|R_0(z)|\psi\rangle}{\mu g(z)} \tag{16}$$

with

$$g(z) = 1 + \frac{1}{\mu} + z\langle\varphi|R_0(z)|\varphi\rangle. \tag{17}$$

Then calculating $R(z)|\psi\rangle$ from (15) and (16) gives the relation

$$R(z) = \left(I - \frac{1}{g(z)}|\varphi\rangle\langle\varphi|\right) R_0(z) - \frac{z}{g(z)} R_0(z)|\varphi\rangle\langle\varphi|R_0(z). \tag{18}$$

To find the spectrum of U we have to examine the singularities of $R(z)$ on the unit circle. Set $z = \mathrm{e}^{-\mathrm{i}(\alpha \pm \mathrm{i}\eta)}$, $\eta > 0$. One has from (17)

$$\begin{aligned} g(\mathrm{e}^{-\mathrm{i}(\alpha\pm\mathrm{i}\eta)}) &= 1 + \frac{1}{\mu} + \frac{1}{T}\int_0^{\theta_0} \mathrm{d}\theta \left|\varphi\left(\frac{\theta}{T}\right)\right|^2 \frac{1}{\mathrm{e}^{-\mathrm{i}(\theta-\alpha\mp\mathrm{i}\eta)} - 1} \qquad \theta_0 = E_0 T \\ &= \frac{\mathrm{i}}{2}\left(\cot\left(\frac{\lambda T}{2}\right) + \frac{1}{T}\int_0^{\theta_0} \mathrm{d}\theta \left|\varphi\left(\frac{\theta}{T}\right)\right|^2 \cot\left(\frac{\theta - \alpha \mp \mathrm{i}\eta}{2}\right)\right) \end{aligned} \tag{19}$$

where the second line results from the formula $\frac{1}{\mathrm{e}^{\mathrm{i}x}-1} = -\frac{1}{2}(1 + \mathrm{i}\cot(\frac{x}{2}))$; $\cot x$ is singular at the points $n\pi$, n integer, and behaves as $(x - n\pi)^{-1}$ near these points.

Suppose that $T < T_0$. Then $R_0(z)$ is holomorphic everywhere except on the sector $0 \leqslant \theta \leqslant \theta_0$ where it has a discontinuity corresponding to the absolutely continuous spectrum of H_0, so formula (18) shows that point spectrum comes from the zeros of $g(z)$. For z on the circle, $z = \mathrm{e}^{-\mathrm{i}\alpha}$ with $\theta_0 < \alpha < 2\pi$, the integrand in (19) has no singularities and $g(z)$ vanishes if α is a solution of the eigenvalue equation

$$h(\alpha) \equiv \frac{1}{T}\int_0^{\theta_0} \mathrm{d}\theta \left|\varphi\left(\frac{\theta}{T}\right)\right|^2 \cot\left(\frac{\theta - \alpha}{2}\right) = -\cot\left(\frac{\lambda T}{2}\right) \qquad \theta_0 < \alpha < 2\pi. \tag{20}$$

Since $\frac{\mathrm{d}}{\mathrm{d}x}\cot x = -\frac{1}{(\sin x)^2}$, the function $h(\alpha)$ defined in (20) is monotonously increasing in the sector $\theta_0 \leqslant \alpha \leqslant 2\pi$ between $h(\theta_0)$ and $h(2\pi) = h(0)$. Hence, it has exactly one solution $\alpha_c = \epsilon_c T$, $\theta_0 < \alpha_c < 2\pi$ if

$$h(\theta_0) < -\cot\left(\frac{\lambda T}{2}\right) < h(2\pi). \tag{21}$$

We distinguish the following cases.

(i) $h(\theta_0) = -\infty$, $h(0) = \infty$. Then (20) has a solution for all $T < T_0$ and all coupling constants λ, provided that $\frac{\lambda T}{2} \neq n\pi$. This case occurs for instance if $\varphi(0) \neq 0$ and $\varphi(E_0) \neq 0$ causing a logarithmic divergence of $h(\alpha)$ at $\alpha = \theta_0$ and $\alpha = 2\pi$.

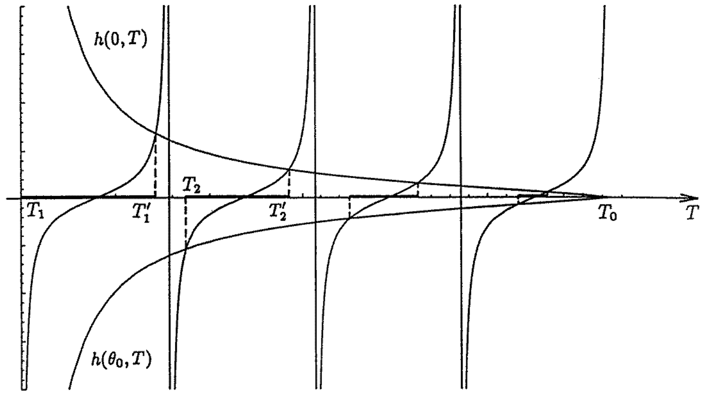

Figure 1. Intervals of stability. Here $E_0 = 1$, $T_0 = 2\pi$, $\varphi(E) = \sqrt{30}E(E-1)$, $\lambda = 4$. One has the symmetry $\varphi(E) = \varphi(1-E)$, $h(\theta_0, T) = -h(0, T)$.

(ii) One of the quantities $h(\theta_0)$ and $h(2\pi) = h(0)$ (or both) are finite. Then for $T < T_0$, equation (20) has a solution if $\frac{\lambda T}{2}$ belongs to a certain range of values defined by condition (21), consisiting in general of an union of semi-infinite or finite intervals (called stability intervals) because of the periodicity of $\cot(\frac{\lambda T}{2})$. In this case $\varphi(E)$ has to vanish at one or both of the points $E = 0$, $E = E_0$. We may assume for instance

$$\begin{aligned}&\varphi(E) \sim \text{constant } E^{\nu} \qquad E \to 0\\ &\varphi(E) \sim \text{constant } (E-E_0)^{\nu} \qquad E \to E_0 \qquad \nu > 0.\end{aligned} \tag{22}$$

Stability intervals are illustrated in figure 1 and discussed in more detail in section 4.

In either of the two cases this produces a single pole in the resolvent (18), and thus an eigenstate ψ_c of U satisfying (13). One verifies that

$$\psi_c(E) = a\frac{\varphi(E)}{\mathrm{e}^{-\mathrm{i}(\epsilon_c T - ET)} - 1} \tag{23}$$

where a is a normalization constant. One also sees that the eigenvalue equation (20) as well as ϵ_c and ψ_c reduce to the corresponding quantities for the static Hamiltonian (14) as $T \to 0$.

If $T < T_0$ but $0 < \alpha < \theta_0$, $\cot(\frac{\theta-\alpha\mp\mathrm{i}\eta}{2}) \sim \frac{2}{\theta-\alpha\mp\mathrm{i}\eta}$ is singular for $\theta = \alpha$ as $\eta \to 0$ and the Cauchy principal value formula gives

$$\lim_{\eta\to 0} g(\mathrm{e}^{-\mathrm{i}(\alpha\pm\mathrm{i}\eta)}) = \mp A(\alpha) + \mathrm{i}B(\alpha) \tag{24}$$

with

$$A(\alpha) = \frac{\pi}{T}\left|\varphi\left(\frac{\alpha}{T}\right)\right|^2 \tag{25}$$

$$B(\alpha) = \frac{1}{2}\left(\cot\left(\frac{\lambda T}{2}\right) + \frac{1}{T}\mathcal{P}\!\!\int_0^{\theta_0} \mathrm{d}\theta \left|\varphi\left(\frac{\theta}{T}\right)\right|^2 \cot\left(\frac{\theta-\alpha}{2}\right)\right). \tag{26}$$

Since $\varphi(E)$ does not vanish on the spectrum of H_0 by assumption, $g(z)$ does not vanish either on the sector $0 < \alpha < \theta_0$. In this sector, $R(z)$ has a discontinuity that can be calculated from (18) in terms of those of $R_0(z)$ and $g(z)$. Since both the spectral measure of U_0 and the function $\varphi(E)$ are supposed absolutely continuous, the same is true for the spectral measure of U in this sector (a discussion of the endpoints $E = 0$ and $E = E_0$ will be given in relation with the resonances in section 4).

If $T > T_0$, one applies the principal value formula at all the singular points $\theta = \alpha + 2n\pi$ occurring in the integrand of (19) and formula (24) still holds with (25) replaced by

$$A(\alpha) = \frac{\pi}{T} \sum_n \left| \varphi\left(\frac{\alpha + 2\pi n}{T}\right) \right|^2 . \tag{27}$$

The spectrum of H_0 wraps the unit circle and $R_0(z)$ is discontinuous on the whole circle: the situation is as that just described above and the spectrum of U is everywhere absolutely continuous.

Therefore, as long as $0 < T < T_0$, we can have a cyclic state (certainly in the case (i) and for appropriate values of λ and T in case (ii)), but cyclic states cannot persist when T becomes larger than T_0.

3. The Lippman–Schwinger equations and the T-matrix

In complete analogy with the scattering theory for time-independent potentials, we establish the equivalent of the Lippman-Schwinger equations for our problem and give an explicit form of the scattering matrix in terms of the resolvent of the Floquet operator.

The following derivation applies to time periodic scattering systems with interacting evolution $U(t_0, t)$ and free evolution $U_0(t_0, t)$ that form a complete scattering system. It is assumed that the wave operators $\Omega_\pm$ are defined as the usual strong limits on $\mathcal{H}$

$$\Omega_\pm = \operatorname*{s-lim}_{t\to\infty} U^*(0, \pm t) U_0(0, \pm t) \tag{28}$$

and are complete in the sense that the ranges $\mathcal{R}(\Omega_\pm)$ of both wave operators are equal, leading to a unitary scattering operator $S = \Omega_+^* \Omega_-$. Let $U = U(0, T)$ and $U_0 = U_0(0, T)$ be the evolution operators over one period. Note that by periodicity and unitarity $U = U(0, T) = U(-T, 0) = U(0, -T)^*$, and the same for $U_0 = U_0(0, -T)^*$, so

$$\Omega_+ = \operatorname*{s-lim}_{n\to\infty} (U^*)^n U_0^n \tag{29}$$

$$\Omega_- = \operatorname*{s-lim}_{n\to\infty} U^n (U_0^*)^n . \tag{30}$$

The desired integral representation of $\Omega_\pm$ is based on the following lemma.

Lemma. Let B_n be a sequence of bounded operators with $\|B_n\| \leqslant 1$ converging strongly to B as $n \to \infty$ and $\xi < 1$; then

$$B = \operatorname*{s-lim}_{\xi\to 1_-} (1-\xi) \sum_{n=0}^{\infty} \xi^n B_n . \tag{31}$$

Proof. For $\psi \in \mathcal{H}$ and $\xi < 1$,

$$\left\| B\psi - (1-\xi) \sum_{n=0}^{\infty} \xi^n B_n \psi \right\| = (1-\xi) \left\| \sum_{n=0}^{\infty} \xi^n (B\psi - B_n\psi) \right\|$$
$$\leqslant (1-\xi) \sum_{n=0}^{N} \|B\psi - B_n\psi\| + (1-\xi) \sum_{N}^{\infty} \xi^n \|B\psi - B_n\psi\| .$$

Choosing N so large that $\|B\psi - B_n\psi\| \leqslant \varepsilon$, $n \geqslant N$, the second term is less than ε uniformly with respect to ξ, and the first term tends to zero as $\xi \to 1$. □

We apply the lemma to the limits (29) and (30)

$$\Omega_+ = \operatorname*{s-lim}_{\xi\to 1_-}(1-\xi)\sum_{n=0}^{\infty}\xi^n (U^*)^n U_0^n$$
$$= \operatorname*{s-lim}_{\xi\to 1_-}(1-\xi)\int_0^{2\pi}(1-\xi U^* \mathrm{e}^{-\mathrm{i}\theta})^{-1}\,\mathrm{d}F_0(\theta)$$
$$= \operatorname*{s-lim}_{\xi\to 1_-}(1-\xi)\int_0^{2\pi}(U-\xi\mathrm{e}^{-\mathrm{i}\theta})^{-1}U\,\mathrm{d}F_0(\theta) \tag{32}$$

where the second line results from the introduction of the spectral representation (8). Thus we find

$$I-\Omega_+ = \operatorname*{s-lim}_{\xi\to 1_-}\int_0^{2\pi}(I-(\xi-1)(U-\xi\mathrm{e}^{-\mathrm{i}\theta})^{-1}U)\,\mathrm{d}F_0(\theta)$$
$$= \operatorname*{s-lim}_{\xi\to 1_-}\int_0^{2\pi}(U-\xi\mathrm{e}^{-\mathrm{i}\theta})^{-1}(U-\mathrm{e}^{-\mathrm{i}\theta})\,\mathrm{d}F_0(\theta). \tag{33}$$

In a similar way, with $\zeta = \frac{1}{\xi} > 1$

$$\Omega_- = \operatorname*{s-lim}_{\xi\to 1_-}(1-\xi)\sum_{n=0}^{\infty}\xi^n U^n (U_0^*)^n$$
$$= \operatorname*{s-lim}_{\zeta\to 1_+}(\zeta-1)\int_0^{2\pi}(\zeta\mathrm{e}^{-\mathrm{i}\theta}-U)^{-1}\mathrm{e}^{-\mathrm{i}\theta}\,\mathrm{d}F_0(\theta) \tag{34}$$

and

$$I-\Omega_- = \operatorname*{s-lim}_{\zeta\to 1_+}\int_0^{2\pi}(I-(\zeta-1)(\zeta\mathrm{e}^{-\mathrm{i}\theta}-U)^{-1}\mathrm{e}^{-\mathrm{i}\theta})\,\mathrm{d}F_0(\theta)$$
$$= \operatorname*{s-lim}_{\zeta\to 1_+}\int_0^{2\pi}(\zeta\mathrm{e}^{-\mathrm{i}\theta}-U)^{-1}(\mathrm{e}^{-\mathrm{i}\theta}-U)\,\mathrm{d}F_0(\theta). \tag{35}$$

Setting $\xi = \mathrm{e}^{-\eta}$, $\zeta = \mathrm{e}^{\eta}$, $\eta > 0$, both relations (33) and (35) can be written in the traditional form of the Lippman–Schwinger equations

$$\Omega_\pm = I - \operatorname*{s-lim}_{\eta\to 0_+}\int_0^{2\pi}R(\mathrm{e}^{-\mathrm{i}(\theta\mp\mathrm{i}\eta)})W\,\mathrm{d}F_0(\theta) \tag{36}$$

where the 'interaction' $W = U - U_0$ is represented by the difference of the Floquet operator and the corresponding free evolution. The integral representation of $\Omega_\pm^*$ is obtained from the fact that on $\mathcal{R} = \mathcal{R}(\Omega_-) = \mathcal{R}(\Omega_+)$

$$\Omega_+^* = \operatorname*{s-lim}_{n\to\infty}(U_0^*)^n U^n$$
$$\Omega_-^* = \operatorname*{s-lim}_{n\to\infty}U_0^n (U^*)^n.$$

Thus, by exchanging the roles of U_0 and U in the derivation (32)–(35), one finds on $\mathcal{R}$

$$\Omega_\pm^* = I + \operatorname*{s-lim}_{\eta\to 0_+}\int_0^{2\pi}R_0(\mathrm{e}^{-\mathrm{i}(\theta\mp\mathrm{i}\eta)})W\,\mathrm{d}F(\theta). \tag{37}$$

Then the S operator is found from the usual manipulations

$$S - I = (\Omega_+^* - \Omega_-^*)\Omega_- = \operatorname*{s-lim}_{\nu\to 0_+}\int_0^{2\pi}\{R_0(\mathrm{e}^{-\mathrm{i}(\theta-\mathrm{i}\nu)}) - R_0(\mathrm{e}^{-\mathrm{i}(\theta+\mathrm{i}\nu)})\}W\Omega_-\,\mathrm{d}F_0(\theta)$$
$$= \operatorname*{s-lim}_{\nu\to 0_+}\operatorname*{s-lim}_{\eta\to 0_+}\int_0^{2\pi}\{R_0(\mathrm{e}^{-\mathrm{i}(\theta-\mathrm{i}\nu)}) - R_0(\mathrm{e}^{-\mathrm{i}(\theta+\mathrm{i}\nu)})\}$$
$$\times\{W - WR(\mathrm{e}^{-\mathrm{i}(\theta+\mathrm{i}\eta)})W\}\,\mathrm{d}F_0(\theta). \tag{38}$$

The second line results from (37) and from the intertwining relation $\mathrm{d}F(\theta)\Omega_- = \Omega_-\,\mathrm{d}F_0(\theta)$. To obtain the third line, we have introduced (36) and used the formal relation $\mathrm{d}F_0(\theta')\,\mathrm{d}F_0(\theta) = \delta(\theta'-\theta)\,\mathrm{d}\theta'\,\mathrm{d}F_0(\theta)$. Representation (38) of the scattering operator defines the $\mathcal{T}$-operator by

$$\mathcal{T}(z) = W - WR(z)W \tag{39}$$

in analogy with its usual definition when the interaction is time independent.

To express the matrix elements of the scattering operator $\langle\epsilon + n\omega|S|\epsilon' + n'\omega\rangle$ in the energy representation we introduce (12) in (38) setting $\theta = \epsilon'' T$

$$S - I = \operatorname*{s-lim}_{\nu\to 0_+}\operatorname*{s-lim}_{\eta\to 0_+}\sum_{n''\geqslant 0}\int_0^{\omega}\mathrm{d}\epsilon''\,\{R_0(\mathrm{e}^{-\mathrm{i}(\epsilon''-\mathrm{i}\nu)}) - R_0(\mathrm{e}^{-\mathrm{i}(\epsilon''+\mathrm{i}\nu)})\}$$
$$\times\mathcal{T}(\mathrm{e}^{-\mathrm{i}(\epsilon''+\mathrm{i}\eta)})|\epsilon'' + n''\omega\rangle\langle\epsilon'' + n''\omega|. \tag{40}$$

Using the fact that in the spectral representation of H_0 the difference of the free resolvents occurring in (40) is the following function of the quasi-energy ϵ

$$\frac{1}{\mathrm{e}^{-\mathrm{i}\epsilon T} - \mathrm{e}^{-\mathrm{i}(\epsilon'' T - \mathrm{i}\nu)}} - \frac{1}{\mathrm{e}^{-\mathrm{i}\epsilon T} - \mathrm{e}^{-\mathrm{i}(\epsilon'' T + \mathrm{i}\nu)}} \to \frac{2\pi}{T}\mathrm{e}^{\mathrm{i}\epsilon T}\delta(\epsilon - \epsilon'') \qquad \text{as } \nu \to 0 \tag{41}$$

one finds that the S-operator can be reduced to the quasi-energy shell matrix $S(\epsilon)$

$$\langle\epsilon + n\omega|S|\epsilon' + n'\omega\rangle = \delta(\epsilon - \epsilon')\langle n|S(\epsilon)|n'\rangle \tag{42}$$

with $\langle n|S(\epsilon)|n'\rangle$ given by

$$\langle n|(S(\epsilon) - I)|n'\rangle = \frac{2\pi\,\mathrm{e}^{\mathrm{i}\epsilon T}}{T}\lim_{\eta\to 0}\langle\epsilon + n\omega|\mathcal{T}(\mathrm{e}^{-\mathrm{i}(\epsilon T + \mathrm{i}\eta)})|\epsilon + n'\omega\rangle \tag{43}$$

with ϵ, n and n' such that both $\epsilon + n\omega$ and $\epsilon + n'\omega$ belong to the spectrum $[0, E_0]$ of H_0.

The derivation presented here is formal, but it can be justified by adapting the proofs of section (6.2) of [10] to our case.

4. Scattering and resonances

We apply the general results of section 3 to our system. From (5) we have $W = \mu U_0|\varphi\rangle\langle\varphi|$ so (39) gives

$$\mathcal{T}(z) = \mu(1 - \mu\langle\varphi|R(z)U_0|\varphi\rangle)U_0|\varphi\rangle\langle\varphi|. \tag{44}$$

But in view of (16) and (17)

$$\mu\langle\varphi|R(z)U_0|\varphi\rangle = \frac{\langle\varphi|R_0(z)U_0|\varphi\rangle}{g(z)}$$
$$= \frac{1 + z\langle\varphi|R_0(z)|\varphi\rangle}{g(z)} = 1 - \frac{1}{\mu g(z)} \tag{45}$$

so that

$$\mathcal{T}(z) = \frac{1}{g(z)}U_0|\varphi\rangle\langle\varphi|. \tag{46}$$

Hence according to (43) and (24) the scattering matrix is

$$\langle n|S(\epsilon)|n'\rangle = \delta_{n,n'} + \frac{2\pi}{T}\left(\frac{\varphi(\epsilon + n\omega)\varphi^*(\epsilon + n'\omega)}{-A(\epsilon T) + \mathrm{i}B(\epsilon T)}\right) \tag{47}$$

with $\epsilon + n\omega$ and $\epsilon + n'\omega$ in $[0, E_0]$. The energy conserving transitions $n = n'$ are referred to as the elastic channel, and the transitions with $n \neq n'$ (corresponding to emission or absorbtion of $(n - n')\omega$ energy quanta) as the inelastic channels.

If $T < T_0$, necessarily $(n, n') = (0, 0)$; only the elastic channel is open and from (25)

$$S_{00}(\epsilon) \equiv \langle n = 0|S(\epsilon)|n' = 0\rangle = \frac{A(\epsilon T) + \mathrm{i}B(\epsilon T)}{-A(\epsilon T) + \mathrm{i}B(\epsilon T)} \qquad 0 \leqslant \epsilon \leqslant E_0 \tag{48}$$

is a pure phase factor.

If $T_0 < T < 2T_0$, in addition to the elastic channel $(n, n') = (0, 0)$, $(1, 1)$, one has the two inelastic channels $(1, 0)$ and $(0, 1)$ allowing for the emission or absorbtion of one quantum ω. As T increases more inelastic channels become open. It is not hard to verify on (47) that the S-matrix satisfies the general unitarity relation

$$\sum_n |\langle n|S(\epsilon)|n'\rangle|^2 = \sum_{n'} |\langle n|S(\epsilon)|n'\rangle|^2 = 1. \tag{49}$$

We now come to the relation of the S-matrix with the cyclic states and the question of the resonances.

First, we verify Levinson's theorem. Suppose that $T < T_0$ and that we have a cyclic state as described in section 2. We define the scattering phase shift $\delta(\epsilon)$ in the elastic channel by

$$S_{00}(\epsilon) = \mathrm{e}^{2\mathrm{i}\delta(\epsilon)} \tag{50}$$

i.e. from (48)

$$\delta(\epsilon) = \arctan\left(\frac{B(\epsilon T)}{A(\epsilon T)}\right) + \frac{\pi}{2} \qquad -\frac{\pi}{2} < \arctan x < \frac{\pi}{2}. \tag{51}$$

In view of (21) one has $B(E_0T) = B(\theta_0) = \frac{1}{2}(\cot(\frac{\lambda T}{2}) + h(\theta_0)) < 0$ and $B(2\pi) = B(0) = \frac{1}{2}(\cot(\frac{\lambda T}{2}) + h(0)) > 0$. Either $h(\theta_0) = -\infty$, or $h(\theta_0)$ is finite and $\varphi(E_0) = 0$. Hence, in both cases $\lim_{\epsilon\to E_0} \frac{B(\epsilon T)}{A(\epsilon T)} = -\infty$, and in the same way, $\lim_{\epsilon\to 0} \frac{B(\epsilon T)}{A(\epsilon T)} = \infty$. This implies

$$\delta(0) - \delta(E_0) = \pi. \tag{52}$$

If there is no cyclic state, the inequality (21) does not hold: $B(E_0T)$ and $B(0)$ have the same sign, and $\frac{B(\epsilon T)}{A(\epsilon T)}$ tends to ∞ (or $-\infty$) in both limits $\epsilon \to E_0$ and $\epsilon \to 0$, leading to

$$\delta(0) - \delta(E_0) = 0. \tag{53}$$

The relations (52) and (53) constitute the Levinson theorem for the present model. The validity of Levinson's theorem for more general time periodic potentials has been established in [11] in the framework of the quasistationary equations.

To discuss resonances in a definite situation, we assume that $\varphi(E)$ is differentiable and that condition (22) in case (ii) of section 2 holds with $\nu > \frac{1}{2}$. Then, writing now explicitly the period dependence in the function $h(\alpha, T)$ (20) we have that

$$\begin{aligned} h(0, T) &= \int_0^{E_0} \mathrm{d}E|\varphi(E)|^2 \cot\left(\frac{ET}{2}\right) \\ h(\theta_0, T) &= \int_0^{E_0} \mathrm{d}E|\varphi(E)|^2 \cot\left(\frac{(E - E_0)T}{2}\right) \qquad \theta_0 = E_0T \end{aligned} \tag{54}$$

are finite for $T > 0$; $h(0, T) \to \infty$ as $T \to 0$ and is decreasing in the interval $0 < T \leqslant T_0$. Likewise $h(\theta_0, T) \to -\infty$ as $T \to 0$ and increases in this interval and both functions coincide at T_0. In view of condition (21), for fixed λ, this determines a number of intervals of stability (T_i, T_i'), $i = 1, 2, \ldots$, where we will have a cyclic state (see figure 1). As T

varies in (T_i, T_i') and $\lambda > 0$, the cyclic state emerges from the continuum threshold θ_0 at T_i and migrates towards the other threshold $2\pi = 0 \pmod{2\pi}$ where it is absorbed at T_i'.

More specifically, consider a period T in the interval (T_i, T_i'), T close to $T_c \equiv T_i'$ and define the function $B(\alpha, T)$ on the whole circle by

$$B(\alpha, T) = \frac{1}{2}\left(\cot\left(\frac{\lambda T}{2}\right) + h(\alpha, T)\right) \tag{55}$$

for $\theta_0 < \alpha < 2\pi$ and by expression (26) for $0 < \alpha < \theta_0$. The conditions above on $\varphi(E)$ imply that $B(\alpha, T)$ is continuously differentiable in α and T for (α, T) in a neighbourhood of $(0, T_c)$ and $\frac{\partial}{\partial\alpha}B(\alpha, T)|_{\alpha=0,T=T_c} > 0$. Then the implicit function theorem ensures that $B(\alpha, T)$ has a differentiable zero $\alpha(T)$ for T in a neighbourhood of T_c and $B(\alpha, T) \sim b(\alpha - \alpha(T))$, $b > 0$. For T below T_c, $\alpha(T) = \alpha_c = \epsilon_c T$ is the eigenphase of the cyclic states (see (20)), whereas for T just above T_c, $\alpha(T) = \alpha_r = \epsilon_r T$ is the location of a resonance with quasi-energy ϵ_r. Indeed, from (48), $|S_{00}(\epsilon) - 1|^2$ (and the cross section which is proportional to it) will have the Breit–Wigner form

$$\begin{aligned} |S_{00}(\epsilon) - 1|^2 &= \frac{2(A(\epsilon T))^2}{(B(\epsilon T))^2 + (A(\epsilon T))^2} \\ &\sim \frac{2\Gamma_r^2}{(\epsilon - \epsilon_r)^2 + \Gamma_r^2} \end{aligned} \tag{56}$$

when T is close to T_c $(T > T_c)$, and $|\epsilon - \epsilon_r| \ll \epsilon_r$. By (25) and (22) the width Γ_r of the resonance (the inverse life time) behaves as

$$\begin{aligned} \Gamma_r = \frac{A(\epsilon_r T)}{bT} &= \frac{\pi}{bT^2}|\varphi(\epsilon_r)|^2 \\ &=\sim \text{constant } \epsilon_r^{2\nu} \sim \text{constant } (T - T_c)^{2\nu} \end{aligned} \tag{57}$$

as $T \to T_c$.

Note that at $T = T_c$ we still have a cyclic state with quasi-energy $\epsilon_c = 0$ that coincides with the threshold of the continuous spectrum. Since the ratio $\frac{B(\epsilon T)}{A(\epsilon T)}$ behaves there as $\epsilon^{1-2\nu} \to \infty$, $\epsilon \to 0$, one still obtains the Levinson relation (52).

All these considerations can of course be reproduced for periods T in the vicinity of T_i when the eigenphase of the cyclic state is close to the other threshold θ_0. The number of intervals of stability giving rise to cyclic states decreases as the coupling constant λ becomes weaker. A special situation is obtained for the value $\lambda = \frac{E_0}{2}$ such that $\cot(\frac{\lambda T}{2})$ has its first zero at T_0. Then a cyclic state can remain present until the closure of the gap at the critical period T_0 and be transformed into a resonance beyond T_0.

If $0 < \nu \leqslant \frac{1}{2}$, the function $B(\alpha, T)$ is still continuous but no more differentiable, and threshold behaviours must be studied in more detail. If $\varphi(E)$ does not vanish at one or the other threshold, we will have no resonance there. Nothing is said here about resonances that may appear far from the thresholds due to particular properties of the function $\varphi(E)$.

For brevity, we have assumed that the spectrum of H_0 is simple. If it has a multiplicity indexed by parameters σ it suffices to replace the (improper) eigenenergy kets $|E\rangle$ by $|E, \sigma\rangle$. The matrix elements of $S(E)$ will be indexed both by σ and the channel indices n, but the results remain the same.

We conclude this section with an example. Consider the scattering of an electron on the one-dimensional lattice $\{ja, j = \ldots, -1, 0, 1, \ldots\}$, with spacing a, by a kicked impurity $\lambda f(t)|\varphi\rangle\langle\varphi|$ located at the origin $j = 0$, $\varphi(j) = \delta_{j,0}$. The free Hamiltonian $H_0 = -\frac{1}{2}\Delta_a$ is the finite difference Laplacian with energy dispersion $E(p) = 1 - \cos p$, $-\frac{\pi}{a} \leqslant p \leqslant \frac{\pi}{a}$.

In the spectral representation $|E, \sigma\rangle$ of H_0 ($\sigma = \pm 1$ corresponding to positive or negative momentum p), one finds

$$\varphi(E, \sigma) = \langle E, \sigma|\varphi\rangle = \frac{1}{\sqrt{2\pi}} \frac{1}{(1-(E-1)^2)^{1/4}}. \tag{58}$$

Here $\varphi(E, \sigma)$ does not vanish at the endpoints $E = 0$ and $E = 2$ of the free spectrum so that we are in case (i) of section 2. There will be a cyclic state for all periods ($\frac{\lambda T}{2} \neq n\pi$) up to $T_0 = \pi$ but it will not give rise to a resonance.

5. Concluding remarks

In this paper, we have analysed a solvable model of a scattering system with a time periodic interaction whose Floquet operator has both continuous and point spectrum. We emphasize that this situation is much more general than that presented by means of this particular example†. Consider a free Hamiltonian H_0 that has absolutely continuous spectrum in $[0, E_0]$ and a bounded smooth time periodic interaction $V(t)$ with period T, and let $U(0, t)$ be the evolution operator associated to the Hamiltonian $H(t) = H_0 + V(t)$. Suppose for instance that, in addition to continuous spectrum in $[0, E_0]$, the time-averaged Hamiltonian $\bar{H} = \frac{1}{T}\int_0^T \mathrm{d}t\, H(t)$ has an isolated non-degenerate eigenvalue $\bar{E}$. Proceeding as in the proof of proposition 2 of [8] (the high-frequency limit), one obtains the operator norm estimate

$$\|U(0, t) - \mathrm{e}^{-\mathrm{i}\bar{H}t}\| = \mathrm{O}(T) \tag{59}$$

provided that t is bounded away from 0. Fix $t > 0$ (small) such that $\mathrm{e}^{-\mathrm{i}\bar{E}t}$ is an isolated eigenvalue of $\mathrm{e}^{-\mathrm{i}\bar{H}t}$ on the unit circle. Then Rellich's theorem ensures that for T small enough, $U(0, t)$ also has an eigenvalue $\mathrm{e}^{-\mathrm{i}\alpha(t)}$ close to it. For t of the form $t = NT$, N integer, we have $(U(0, T))^N = U(0, NT))$, and this implies that the Floquet operator $U(0, T)$ has also an eigenvalue $\mathrm{e}^{-\mathrm{i}\alpha_c}$ with $\alpha(NT) = N\alpha_c$ In particular, the qualitative findings of this paper remain true if $f(t)$ (2) is replaced by a smooth periodic function of t.

This shows that this mechanism gives rise to cyclic states of the Floquet operator and is very general, provided that the averaged Hamiltonian $\bar{H}$ has eigenvalues and the spectrum of H_0 is bounded. If the spectrum of H_0 consists of a finite number of bands, cyclic states can appear in all the gaps at high frequency. If H_0 is unbounded, a rank 1 perturbation may induce qualitative changes: for instance the discrete spectrum of the free rotator may be transformed into a singular continuous one [13, 14]. In our class of models, if the spectrum of H_0 is not bounded, say extends on $[0, \infty)$, $U(0, T)$ will also have an absolutely continuous spectrum on the circle, but without gap however short the period may be. Then it is likely that the eigenstates of $\bar{H}$ are turned into resonances of the Floquet operator for $T > 0$. It is an open question to know if the Floquet operator can have other types of cyclic states embedded in its absolutely continuous spectrum. We plan to come to these problems in future work.

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